Selected Solutions to
Steven R. Lay’s
Analysis with an Introduction to Proof

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Chapter 1

Exercise 1.3.7: Suppose $p$ and $q$ are integers. Prove the following.

(b) If $p$ is odd and $q$ is odd, then $pq$ is odd.

Proof: Since $p$ and $q$ are odd, we may write $p = 2k + 1$ and $q = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$. Then observe

$$pq = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1.$$ 

Note that $2k\ell + k + \ell$ is an integer; call it $x$. Then $pq = 2x + 1$ for some $x \in \mathbb{Z}$, so $pq$ is odd. □

(c) If $p$ is odd and $q$ is odd, then $p + 3q$ is even.

Proof: Since $p$ and $q$ are odd, we may write $p = 2k + 1$ and $q = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$. Then observe

$$p + 3q = (2k + 1) + 3(2\ell + 1) = 2k + 1 + 6\ell + 3 = 2k + 6\ell + 4 = 2(k + 3\ell + 2).$$ 

Note that $k + 3\ell + 2$ is an integer; call it $x$. Then $p + 3q = 2x$ for some $x \in \mathbb{Z}$, so $pq$ is even. □

(d) Show if $p$ is odd and $q$ is even, then $p + q$ is odd.

Proof: Since $p$ is odd and $q$ is even, we may write $p = 2k + 1$ and $q = 2\ell$ for some $k, \ell \in \mathbb{Z}$. Then observe

$$p + q = (2k + 1) + (2\ell) = 2k + 2\ell + 1 = 2(k + \ell) + 1.$$ 

Note that $k + \ell$ is an integer; call it $x$. Then $p + q = 2x + 1$ for some $x \in \mathbb{Z}$, so $pq$ is odd. □

(g) If $pq$ is odd, then $p$ is odd and $q$ is odd.

Proof: We proceed contrapositively, so without loss of generality say that at least $p$ is even. Then $p = 2k$ for some $k \in \mathbb{Z}$, and

$$pq = (2k)q = 2(kq).$$ 

Note that $kq$ is an integer, call it $x$. Then $pq = 2x$ for some $x \in \mathbb{Z}$, so $pq$ is even. As this proves the contrapositive, the original statement is also shown to be true. □

Exercise 1.4.3: Prove: For every $\epsilon > 0$, there exists a $\delta > 0$ such that $2 - \delta < x < 2 + \delta$ implies that $11 - \epsilon < 3x + 5 < 11 + \epsilon$. 

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Proof: Let $\epsilon > 0$ and set $\delta = \frac{\epsilon}{3}$. Then starting with the given inequality we may observe the following:

$$2 - \delta < x < 2 + \delta \implies 2 - \frac{\epsilon}{3} < x < 2 + \frac{\epsilon}{3} \implies 6 - \epsilon < 3x < 6 + \epsilon \implies 11 - \epsilon < 3x + 5 < 11 + \epsilon.$$ 

As this was the desired result, we are finished. □

**Exercise 1.4.4:** Prove: For every $\epsilon > 0$, there exists a $\delta > 0$ such that $1 - \delta < x < 1 + \delta$ implies that $2 - \epsilon < 7 - 5x < 2 + \epsilon$.

*Proof:* Let $\epsilon > 0$ and set $\delta = \frac{\epsilon}{5}$. Then starting with the given inequality we may observe the following:

$$1 - \delta < x < 1 + \delta \implies 1 - \frac{\epsilon}{5} < x < 1 + \frac{\epsilon}{5} \implies -5 + \epsilon > -5x > -5 - \epsilon \implies 2 - \epsilon < 7 - 5x < 2 + \epsilon.$$ 

As this was the desired result, we are finished. □

**Exercise 1.4.8:** Prove: If $n$ is odd, then $n^2 = 8k + 1$ for some integer $k$.

*Proof:* Since $n$ is odd, we can write $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$. Then

$$n^2 = (2\ell + 1)^2 = 4\ell^2 + 4\ell + 1 = 4(\ell^2 + \ell) + 1.$$ 

We claim that $\ell^2 + \ell$ is even. To see this, we consider two cases.

Case 1: $\ell$ is odd. Then $\ell^2$ is odd, so $\ell^2 + \ell$ is even.

Case 2: $\ell$ is even. Then $\ell^2$ is even, so $\ell^2 + \ell$ is even.

As the claim holds, we may write $\ell^2 + \ell = 2k$ for some $k \in \mathbb{Z}$. Then it follows that

$$n^2 = 4(\ell^2 + \ell) + 1 = 4(2k) + 1 = 8k + 1,$$

as desired. □

**Exercise 1.4.9:** Prove: There exists an integer $n$ such that $n^2 + 3n/2 = 1$. Is this integer unique?

*Proof:* Observe that

$$n^2 + 3n/2 = 1 \iff 2n^2 + 3n - 2 = 0.$$
Then the quadratic formula gives

\[
n = \frac{-3 \pm \sqrt{9 - 4(2)(-2)}}{2(2)} = -2, \frac{1}{2}
\]
as solutions. But then these are the only possible solutions, so we know \( n = -2 \) is the only integer solution of \( n^2 + 3n/2 = 1 \). □

**Exercise 1.4.22:** Prove or give a counterexample: If \( x \) is irrational, then \( \sqrt{x} \) is irrational.

**Proof:** This is true, and we will prove it contrapositively. That is, we want to show that \( \sqrt{x} \in \mathbb{Q} \) implies \( x \in \mathbb{Q} \).

To that end, say that \( \sqrt{x} \in \mathbb{Q} \). Then for some \( a, b \in \mathbb{Z} \) we may write \( \sqrt{x} = \frac{a}{b} \). As a result, \( x = \frac{a^2}{b^2} \). Since \( a, b \in \mathbb{Z} \), then \( a^2, b^2 \in \mathbb{Z} \), so \( x \in \mathbb{Q} \), as desired. □
Chapter 2

Exercise 2.1.10: Fill in the blanks in the proof of the following theorem.

Proof: Suppose that \( A \subseteq B \). If \( x \in A \cup B \), then \( x \in A \) or \( x \in B \). Since \( A \subseteq B \), in either case we have \( x \in B \). Thus, \( A \cup B \subseteq B \). On the other hand, if \( x \in B \) then \( x \in A \cup B \), so \( B \subseteq A \cup B \). Hence \( A \cup B = B \).

Conversely, suppose that \( A \cup B = B \). If \( x \in A \), then \( x \in A \cup B \). But \( A \cup B = B \), so \( x \in B \). Thus, \( A \subseteq B \). \(\square\)

Exercise 2.1.19: Prove: If \( U = A \cup B \) and \( A \cap B = \emptyset \), then \( A = U \setminus B \).

Proof: Assume that \( U = A \cup B \) and \( A \cap B = \emptyset \).

Let \( x \in A \). Then \( x \in A \cup B = U \). Also \( x \notin B \) since \( A \cap B = \emptyset \). Then \( x \in U \setminus B \).

Now let \( x \in U \setminus B \). Then \( x \in U = A \cup B \) but \( x \notin B \). Hence \( x \in A \).

Then we have \( A = U \setminus B \) by double containment. \(\square\)

Exercise 2.1.20: Prove \( A \cap B \) and \( A \setminus B \) are disjoint and \( A = (A \cap B) \cup (A \setminus B) \).

Proof: We begin by showing the disjointedness. Say that \( x \in A \cap B \). Then \( x \in A \) and \( x \in B \). Hence \( x \notin A \setminus B \). So \( (A \cap B) \cap (A \setminus B) = \emptyset \).

We now show the second part. Say \( x \in A \). Then \( x \in B \) or \( x \notin B \), so we have two cases.

Case 1: \( x \in B \). Then \( x \in A \cap B \), so \( x \in (A \cap B) \cup (A \setminus B) \).

Case 2: \( x \notin B \). Then \( x \in A \setminus B \), so \( x \in (A \cap B) \cup (A \setminus B) \).

Hence, we have \( A \setminus (A \cap B) \cup (A \setminus B) \).

Now say that \( x \in (A \cap B) \cup (A \setminus B) \). Then \( x \in A \cap B \) implies \( x \in A \) or \( x \in A \setminus B \) implies \( x \in A \). So \( (A \cap B) \cup (A \setminus B) \subseteq A \).

Thus, we have \( A = (A \cap B) \cup (A \setminus B) \) by double containment. \(\square\)

Exercise 2.2.9: Fill in the blanks in the proof of the following theorem.

Proof: Let \((x, y) \in (A \cap B) \times C \). Then \( x \in A \cap B \) and \( y \in C \). Since \( x \in A \cap B \), \( x \in A \) and \( x \in B \). Thus \((x, y) \in A \times C \) and \((x, y) \in B \times C \). Hence \((x, y) \in (A \times C) \cap (B \times C) \), so \((A \cap B) \times C \subseteq (A \times C) \cap (B \times C) \).

On the other hand, suppose that \((x, y) \in (A \times C) \cap (B \times C) \). Then \((x, y) \in (A \times C) \) and
\((x, y) \in (B \times C)\). Since \((x, y) \in A \times C\), \(x \in A\) and \(y \in C\). Since \((x, y) \in B \times C\), \(x \in B\) and \(y \in C\). Thus \(x \in A \times B\), so \((x, y) \in (A \cap B) \times C\) and \((A \times C) \cap (B \times C) \subseteq (A \cap B) \times C\). \(\square\)

**Exercise 2.2.10:** Fill in the blanks in the proof of the following theorem.
(b) \((A \cup B) \times C = (A \times C) \cup (B \times C)\)

**Proof:** Say \((x, y) \in (A \cup B) \times C\). Then \(x \in A\) or \(x \in B\) and \(y \in C\). So then \((x, y) \in A \times C\) or \((x, y) \in B \times C\). So then \((x, y) \in (A \times C) \cup (B \times C)\). \((A \cup B) \times C \subseteq (A \times C) \cup (B \times C)\).

Now say \((x, y) \in (A \times C) \cup (B \times C)\). Then \((x, y) \in A \times C\) or \((x, y) \in B \times C\). Then \(x \in A\) or \(x \in B\). Hence \(x \in A \cup B\), so \((x, y) \in (A \cup B) \times C\), and \((A \times C) \cup (B \times C) \subseteq (A \cup B) \times C\).

Thus, we have \((A \cup B) \times C = (A \times C) \cup (B \times C)\) by double containment. \(\square\)

(c) \((A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)\)

**Proof:** Say that \((x, y) \in (A \times B) \cap (C \times D)\). Then \((x, y) \in (A \times B)\) and \((x, y) \in (C \times D)\). So \(x \in A\) and \(x \in C\), and \(y \in B\) and \(y \in D\). So \(x \in A \cap C\) and \(y \in B \cap D\). Hence \((x, y) \in (A \cap C) \times (B \cap D)\), so \((A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)\).

Now say that \((x, y) \in (A \cap C) \times (B \cap D)\). Then \(x \in A\) and \(x \in C\), and \(y \in B\) and \(y \in D\). So then \((x, y) \in A \times B\) and \((x, y) \in C \times D\). Hence \((x, y) \in (A \times B) \cap (C \times D)\), so \((A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)\).

Thus, we have \((A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)\) by double containment. \(\square\)

**Exercise 2.2.16:** Let \(S\) be the Cartesian coordinate plane \(\mathbb{R} \times \mathbb{R}\) and define the equivalence relation \(\mathcal{R}\) on \(S\) by \((a, b)\mathcal{R}(c, d)\) if and only if \(b - 3a = d - 3c\).
(a) Find the partition \(\mathcal{P}\) determined by \(\mathcal{R}\) by describing the pieces in \(\mathcal{P}\).

**Proof:** Essentially, we want to describe the equivalence classes. They will be diagonal lines. In particular, they will be of the form \(y - 3x = k\). The choice of \(k\) determines the line, but equivalent points will be on the same line. \(\square\)

(b) Describe the piece of the partition that contains the point \((2, 5)\).

**Proof:** The piece containing \((2, 5)\) will come from the line with \(k = 5 - 3(2) = -1\). So then the line representing this equivalence class will be given by the equation \(y - 3x = -1\). \(\square\)

**Exercise 2.3.15:** Suppose that \(f : A \to B\). Let \(C_1, C_2 \subseteq A\) and \(D, D_1, D_2 \subseteq B\). Prove the following statements.
(b) \(f [f^{-1}(D)] \subseteq D\).
Proof: Let \( y \in f(f^{-1}(D)) \), then there exists some \( x \in f^{-1}(D) \) such that \( f(x) = y \). Then there exists some \( y' \in D \) such that \( f(x) = y' \). But then \( y = y' \), so \( y \in D \) and \( f(f^{-1}(D)) \subseteq D \). □

(d) \( f(C_1 \cup C_2) = f(C_1) \cup f(C_2) \).

Proof: Let \( y \in f(C_1 \cup C_2) \). Then there exists \( x \in C_1 \cup C_2 \) with \( f(x) = y \). Without loss of generality, say that \( x \in C_1 \). Then \( y \in f(C_1) \), so \( f(C_1 \cup C_2) \subseteq f(C_1) \cup f(C_2) \).

Now say that \( y \in f(C_1) \cup f(C_2) \). Without loss of generality say \( y \in f(C_1) \), then there exists some \( x \in C_1 \) such that \( f(x) = y \). Then \( x \in C_1 \cap C_2 \), so \( y \in f(C_1 \cup C_2) = f(C_1) \cup f(C_2) \).

Then we have \( f(C_1 \cup C_2) = f(C_1) \cup f(C_2) \) by double containment. □

(e) \( f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2) \).

Proof: Let \( x \in f^{-1}(D_1 \cap D_2) \). Then there exists \( y \in D_1 \cap D_2 \) such that \( f(x) = y \). Since \( y \in D_1 \), \( x \in f^{-1}(D_1) \). Similarly, \( x \in f^{-1}(D_2) \). Hence, \( x \in f^{-1}(D_1) \cap f^{-1}(D_2) \), so \( f^{-1}(D_1 \cap D_2) \subseteq f^{-1}(D_1) \cap f^{-1}(D_2) \).

Now say \( x \in f^{-1}(D_1) \cap f^{-1}(D_2) \). Then there exists \( y \in D_1 \) such that \( f(x) = y \). Since \( f \) is a function and \( x \in f^{-1}(D_2) \), we also have \( y \in D_2 \), so \( y \in D_1 \cap D_2 \). Then \( x \in f^{-1}(D_1 \cap D_2) \), so \( f^{-1}(D_1) \cap f^{-1}(D_2) \subseteq f^{-1}(D_1 \cap D_2) \).

Then we have \( f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2) \) by double containment. □

(g) \( f^{-1}(B \setminus D) = A \setminus f^{-1}(D) \).

Proof: Let \( x \in f^{-1}(B \setminus D) \). Then there exists \( y \in B \setminus D \) such that \( f(x) = y \). Note that \( y \notin D \), so \( x \notin f^{-1}(D) \). But as \( x \in A \), we know \( x \in A \setminus f^{-1}(D) \), and \( f^{-1}(B \setminus D) \subseteq A \setminus f^{-1}(D) \).

Now say \( x \in A \setminus f^{-1}(D) \). Note that then \( f(x) \in B \). However, \( x \notin f^{-1}(D) \) implies \( f(x) \notin D \). So then \( f(x) \in B \setminus D \), which implies that \( x \in f^{-1}(B \setminus D) \), and \( A \setminus f^{-1}(D) \subseteq f^{-1}(B \setminus D) \).

Then we have \( f^{-1}(B \setminus D) = A \setminus f^{-1}(D) \) by double containment. □

Exercise 2.3.18: Suppose that \( f : A \to B \). Let \( D \subseteq B \). Prove the following statements.

(b) If \( f \) is surjective, then \( f[f^{-1}(D)] = D \).

Proof: We already know that \( f(f^{-1}(D)) \subseteq D \) from Exercise 2.3.15(b).

Say that \( y \in D \). Then there exists some \( x \in f^{-1}(D) \) such that \( f(x) = y \) because \( f \) is surjective. Then note \( f(x) \in f(f^{-1}(D)) \), so \( D \subseteq f(f^{-1}(D)) \).

Then we have \( f[f^{-1}(D)] = D \) by double containment. □
Exercise 2.3.19: Suppose that $f: A \to B$ and $g: B \to C$ are both injective. Prove that $g \circ f: A \to C$ is injective.

Proof: Let $x_1, x_2 \in A$, and say $g \circ f(x_1) = g \circ f(x_2)$. As $g$ is injective, we know that $f(x_1) = f(x_2)$. But $f$ is also injective, which implies $x_1 = x_2$. Hence, we can conclude $g \circ f(x_1) = g \circ f(x_2)$ implies $x_1 = x_2$, so $g \circ f$ is injective. □

Exercise 2.4.4: (a) Suppose that $m < n$. Prove that the intervals $(0,1)$ and $(m,n)$ are equinumerous by finding a specific bijection between them.

Proof: A bijection will be given by the function $f: (0,1) \to (m,n)$ defined by $f(x) = (n-m)x + m$. (As $f$ is a linear function, its bijectivity is immediate.) □

(b) Use part (a) to prove that any two open intervals are equinumerous.

Proof: By (a) we know that $(a,b)$ is equinumerous to $(0,1)$. Similarly, $(c,d)$ is equinumerous to $(0,1)$. As equinumerity is an equivalence relation, we are finished. □
Chapter 3

**Exercise 3.1.3**: Prove that \(1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)\) for all \(n \in \mathbb{N}\).

**Proof**: Base case: Observe for \(n = 1\) that \(1^2 = 1\) and \(\frac{1}{6}(1)(1+1)(2(1)+1) = \frac{1}{6}6 = 1\). So the base case holds.

Inductive step: Assume that the given statement holds for \(n = k\); we want to show it also holds for \(k + 1\). Now observe the following:

\[
\begin{align*}
1^2 + 2^2 + \cdots + k^2 &= \frac{1}{6}k(k + 1)(2k + 1) \quad \text{Inductive hypothesis} \\
1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 &= \frac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2 \\
&= \frac{1}{6}k(k + 1)(2k + 1) + \frac{1}{6}(k + 1)6(k + 1) \quad \text{Add } (k + 1)^2 \\
&= \frac{1}{6}(k + 1)[k(2k + 1) + 6(k + 1)] \quad \text{Expand} \\
&= \frac{1}{6}(k + 1)[2k^2 + 7k + 6] \quad \text{Factor} \\
&= \frac{1}{6}(k + 1)(k + 2)(2(k + 1) + 1) \quad \text{Combine like terms} \\
\end{align*}
\]

As a result, we have

\[
1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 = \frac{1}{6}(k + 1)(k + 2)(2(k + 1) + 1),
\]

so we are done. □

**Exercise 3.1.6**: Prove that

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}
\]

for all \(n \in \mathbb{N}\).

**Proof**: Base case: Observe for \(n = 1\) that \(\frac{1}{1(2)} = \frac{1}{2}\) and \(\frac{1}{1+1} = \frac{1}{2}\). So the base case holds.

Inductive step: Assume that the given statement holds for \(n = k\); we want to show it also holds for \(k + 1\). Now observe the following:

\[
\begin{align*}
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k + 1)} &= \frac{k}{k + 1} \quad \text{Inductive hypothesis} \\
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k + 1)} + &\frac{1}{(k + 1)(k + 2)} = \frac{k}{k + 1} + \frac{1}{(k + 1)(k + 2)} \quad \text{Add to both sides} \\
&= \frac{k^2 + 2k + 1}{(k + 2)(k + 1)} \quad \text{Combine fractions}
\end{align*}
\]

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Thus, we have that
\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}
\]
so we are done. □

**Exercise 3.1.14:** Prove that $9^n - 4^n$ is a multiple of 5 for all $n \in \mathbb{N}$.

*Proof:* Base case: Observe for $n = 1$ that $9^1 - 4^1 = 5$ and 5 is a multiple of itself.

Inductive step: Assume that the given statement holds for $n = k$; we want to show it also holds for $k + 1$. Now observe that
\[
9^{k+1} - 4^{k+1} = 9 \cdot 9^k - 4 \cdot 4^k = (5 + 4)9^k - 4 \cdot 4^k = 5 \cdot 9^k + 4 \cdot 9^k - 4 \cdot 4^k = 5 \cdot 9^k + 4(9^k - 4^k).
\]
Obviously $5 \cdot 9^k$ is a multiple of 5. Since we also assumed that $9^k - 4^k$ is a multiple of 5, it follows that $5 \cdot 9^k + 4(9^k - 4^k)$ is a multiple of 5, so then $9^{k+1} - 4^{k+1}$ is a multiple of 5. □

**Exercise 3.1.15:** Prove that $12^n - 5^n$ is a multiple of 7 for all $n \in \mathbb{N}$.

*Proof:* Base case: Observe for $n = 1$ that $12^1 - 5^1 = 7$ and 7 is a multiple of itself.

Inductive step: Assume for $n = k$ that $12^k - 5^k$ is a multiple of 7; we want to show it also holds for $k + 1$. Now observe that
\[
12^{k+1} - 5^{k+1} = 12 \cdot 12^k - 5 \cdot 5^k = (7 + 5)12^k - 5 \cdot 4^k = 7 \cdot 12^k + 5 \cdot 12^k - 5 \cdot 5^k = 7 \cdot 12^k + 5(12^k - 5^k).
\]
Obviously $7 \cdot 12^k$ is a multiple of 7. Since we also assumed that $12^k - 5^k$ is a multiple of 7, it follows that $7 \cdot 12^k + 5(12^k - 5^k)$ is a multiple of 7, so then $12^{k+1} - 5^{k+1}$ is a multiple of 7. □

**Exercise 3.1.23:** Indicate for which natural numbers $n$ the given inequality is true. Prove your answers by induction.

(a) $n^2 \leq n!$.

*Proof:* We will show this for $n \geq 4$.

Base case: Observe for $n = 4$ that $4^2 = 16 \leq 24 = 4!$, so the base case holds.

Inductive step: Assume that the given statement is true for $n = k$; we want to show it also holds for $k + 1$. Since $k > 1$, we know that $k + 1 < k^2$; then multiplying by $(k + 1)$ gives
\((k+1)^2 \leq k^2(k+1)\) Multiplying our inductive hypothesis by \((k+1)\) gives \(k^2(k+1) \leq k!(k+1)\). We can then combine these two inequalities to see that
\[
(k + 1)^2 \leq k^2(k + 1) \leq k!(k + 1) = (k + 1)!.
\]
Thus, \((k + 1)^2 \leq (k + 1)!\), as desired. \(\square\)

(c) \(2^n \leq n!\)

**Proof:** We will show this for \(n \geq 4\).

Base case: Observe for \(n = 4\) that \(2^4 = 16 \leq 24 = 4!\), so the base case holds.

Inductive step: Assume that the given statement is true for \(n = k\); we want to show it also holds for \(k + 1\). As \(k > 1\), we know \(k + 1 \geq 2\); combining this with our inductive hypothesis then gives
\[
2^{k+1} = 2^k \cdot 2 \leq 2^k(k + 1) \leq k!(k + 1) = (k + 1)!.
\]
Thus, \(2^{k+1} \leq (k + 1)!\), as desired. \(\square\)

**Exercise 3.2.5:** Prove that \(|xy| = |x| \cdot |y|\).

**Proof:** We have three cases.

Case 1: \(x, y \geq 0\). Then \(xy \geq 0\) implies \(|xy| = xy\). Also \(|x| = x\) and \(|y| = y\), so \(|x||y| = xy\). Hence \(|xy| = xy = |x||y|\).

Case 2: \(x, y \leq 0\). Then \(xy \geq 0\) implies \(|xy| = xy\). Also \(|x| = -x\) and \(|y| = -y\), so \(|x||y| = (-x)(-y) = xy\). Hence \(|xy| = xy = |x||y|\).

Case 3: Without loss of generality, \(x < 0\) and \(y \geq 0\). Then \(xy \leq 0\) implies \(|xy| = -xy\). Also \(|x| = -x\) and \(|y| = y\), so \(|x||y| = -xy\). Hence \(|xy| = -xy = |x||y|\).

As each case holds, we are done. \(\square\)

**Exercise 3.2.6:** (a) Prove \(||x| - |y|| \leq |x - y|\)

**Proof:** Observe the following:
\[
|x| = |x - y + y| \leq |x - y| + |y| \quad \text{Triangle inequality}
\]
\[
\Rightarrow |x| - |y| \leq |x - y| \quad \text{Subtract \(|y|\)}
\]

Similarly, we can derive
\[
|y| - |x| \leq |y - x| = |x - y|
\]
But then
\[ |x| - |y| \leq |x - y| \quad \text{and} \quad |y| - |x| \leq |x - y| \]
implies that
\[ ||x| - |y|| \leq |x - y|, \]
as desired. □

(b) Prove: If \(|x - y| < c\), then \(|x| < |y| + c\).

Proof: Observe from (a) that
\[ c < |x - y| \geq ||x| - |y|| \geq |x| - |y| \implies c > |x| - |y| \implies c + |y| > |x|, \]
as desired. □

Exercise 3.2.7: Suppose that \(x_1, x_2, \ldots, x_n\) are real numbers. Prove that
\[ |x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|. \]

Proof: Base case: When \(n = 2\) we immediately have
\[ |x_1 + x + 2| \leq |x_1| + |x_2| \]
via the triangle inequality.

Inductive step: Assume the given statement holds for \(n = k\); we want to show it also holds for \(k + 1\). Now observe the following:
\[ |x_1 + \cdots + x_k + x_{k+1}| \leq |x_1 + \cdots + x_k| + |x_{k+1}| \quad \text{Triangle inequality} \]
\[ \leq |x_1| + \cdots + |x_k| + |x_{k+1}| \quad \text{Inductive hypothesis} \]

So then we do indeed have
\[ |x_1 + \cdots + x_{k+1}| \leq |x_1| + \cdots + |x_{k+1}|, \]
which completes the induction, so the desired statement holds for all \(n\). □

Exercise 3.3.5: Let \(S\) be a nonempty bounded subset of \(\mathbb{R}\) and let \(m = \sup S\). Prove that \(m \in S\) iff \(m = \max S\).

Proof: (⇒) Assume that \(m \in S\). Because \(m = \sup S\), \(m\) is an upper bound. By definition, a maximum is an upper bound in the set. So then we may conclude that \(m = \max S\).
Now assume that $m = \max S$. Then $m \in S$ by the definition of a maximum. 

Exercise 3.3.6: (a) Let $S$ be a nonempty bounded subset of $\mathbb{R}$. Prove that $\sup S$ is unique.

Proof: By way of contradiction say $n \neq m$ but $n$ and $m$ are both supremum of $S$. Then we know that both are least upper bounds for $S$. Without loss of generality say $m < n$. Then we know that $m \geq s$ and $n \geq s$ for all $s \in S$. But $n$ cannot be the least upper bound because $n > m \geq s$ for all $s \in S$, so we have a contradiction. As a result, we know that $n = m$, so $\sup S$ must be unique. □

(b) Suppose that $m$ and $n$ are both maxima of a set $S$. Prove that $m = n$.

Proof: By way of contradiction, suppose that $n \neq m$ and without loss of generality say $n > m$. Then since $m$ is a maximum, we know that $m \in S$ and $s \leq m$ for all $s \in S$. Note that $n \in S$ because $n$ is a maximum. But $n \nleq m$ and $n \in S$, so this contradicts the fact that $m$ is a maximum. Hence, $n = m$, so the maximum must be unique. □

Exercise 3.3.7: Let $S$ be a nonempty bounded subset of $\mathbb{R}$ and let $k \in \mathbb{R}$. Define $kS = \{ks \mid s \in S\}$. Prove the following:

(b) If $k < 0$, then $\sup(kS) = k \cdot \inf S$ and $\inf(kS) = k \cdot \sup S$.

Proof: Let $m = \inf S$. Then $m \leq s$ for all $s \in S$, so multiplying by $k$ gives $km \geq ks$ for all $ks \in kS$. Hence, we have that $km$ is an upper bound of $kS$. We still need to show that $km$ is a least upper bound of $kS$.

By way of contradiction, say that there exists $x \in \mathbb{R}$ with $x < km$ and $x \geq ks$ for all $ks \in kS$. Then dividing by $k$ gives $\frac{x}{k} > m$ and $\frac{x}{k} \leq s$ for all $s \in S$. But then this contradicts the fact that $m$ is the infimum of $S$, so we know that such an $x$ must not exist. As a result, $km$ is indeed the least upper bound of $kS$, so $\sup(kS) = k \cdot \inf S$, as desired.

Now let $M = \sup S$. Then $M \geq s$ for all $s \in S$ so multiplying by $k$ gives $kM \leq ks$ for all $ks \in kS$. Hence, we have that $kM$ is a lower bound of $kS$. We still need to show that $kM$ is a greatest lower bound of $kS$.

By way of contradiction, say that there exists $x \in \mathbb{R}$ with $x > kM$ and $x \leq ks$ for all $ks \in kS$. Then dividing by $k$ gives $\frac{x}{k} < M$ and $\frac{x}{k} \geq s$ for all $s \in S$. But then this contradicts the fact that $M$ is the supremum of $S$, so we know that such an $x$ must not exist. As a result, $kM$ is indeed the greatest lower bound of $kS$, so $\inf(kS) = k \cdot \sup S$, as desired. □

Exercise 3.3.9: (a) Prove: If $y > 0$, then there exists $n \in \mathbb{N}$ such that $n - 1 \leq y < n$. 

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Proof: Let $S = \{x \in \mathbb{N} \mid x > y\}$. By the well-ordering property, there exists some minimum in the set $S$, call it $n$. This means that $n - 1 \not\in y$, so $n - 1 \leq y$. Hence $n - 1 \leq y < n$. □

(b) Prove that the $n$ in part (a) is unique.

Proof: By way of contradiction suppose there is some $m \neq n$ such that $m - 1 \leq y < m$. Since $m > y$, $m \in S$ (where $S$ is as defined in (a)). As $n = \min S$, we know that $m > n$, so $m \geq n + 1$. But then $m - 1 \geq n$ and this contracts $m - 1 \leq y < m$ because $n > y$. Hence $m = n$, so we may conclude that $n$ is unique. □

Exercise 3.3.10: (b) Prove that if $x$ and $y$ are real numbers with $x < y$, then there are infinitely many irrational numbers in the interval $[x, y]$.

Proof: By way of contradiction, suppose that there are only finite many irrational number in $[x, y]$, call that number $n$. We will show that there are at least $n + 1$ such irrationals, a contradiction.

As $x < y$, we may apply Theorem 3.3.15 to obtain an irrational $w_0$ such that $x < w_0 < y$. Then since $x < w_0$, we may apply the theorem again to obtain an irrational $w_1$ such that $x < w_1 < w_0$. Continue this process until $w_n$ is obtained, and then note that the set $\{w_0, \ldots, w_n\}$ contains $n + 1$ irrational numbers in $[x, y]$, which is the desired contradiction.

Hence, we know that our assumption was incorrect, so there must indeed be infinitely many irrational numbers in $[x, y]$. □

Exercise 3.4.13: Prove: $(\text{cl } S) \setminus (\text{int } S) = \text{bd } S$.

Proof: Let $x \in (\text{cl } S) \setminus (\text{int } S)$. Then $x \in S \cup \text{bd } S$ but $x \not\in \text{int } S$. If $x \in \text{bd } S$ we are done, so say $x \in S$. Since $x \not\in \text{int } S$, we know any open $N$ about $x$ will be such that $N \not\subseteq S$. Combining this with the fact that $x \in S$ then gives $N \cap S \neq \emptyset$ and $N \cap (\mathbb{R} \setminus S) \neq \emptyset$ for all $N$, so $x \in \text{bd } S$. Hence, $(\text{cl } S) \setminus (\text{int } S) \subseteq \text{bd } S$.

Now say $x \in \text{bd } S$. Then for all open $N$ about $x$, $N \cap S \neq \emptyset$ and $N \cap (\mathbb{R} \setminus S) = \emptyset$. The first staement tells us that $x \in \text{cl } S$ and the second that $x \not\in \text{int } S$. Hence, $x \in (\text{cl } S) \setminus (\text{int } S)$. □

Exercise 3.4.14: Let $S$ be a bounded infinite set and let $x = \sup S$. Prove: If $x \not\in S$, then $x \in S'$.

Proof: By way of contradiction suppose that $x \not\in S'$. Then there exists an open $N$ such that $x \in N$ and $N \cap S = \emptyset$. Then there exists some $\delta$-ball $B_\delta(x) \subseteq N$ with $\delta > 0$. Note that as $x = \sup S$, $x \geq s$ for all $s \in S$. As $N \cap S = \emptyset$, this mean that $n \geq s$ for all $s \in S$, $n \in N$. But
then \( x - \frac{\delta}{2} \in N \) and \( s \leq x - \frac{\delta}{2} < x \) for all \( s \in S \), which contradicts the fact that \( x = \sup S \). □
Chapter 4

**Exercise 4.1.6:** (e) Show \( \lim_{n \to \infty} \frac{n+3}{n^2-13} = 0. \)

*Proof:* Let \( \epsilon > 0 \) and \( N > \max\{6, \frac{8}{\epsilon}\} \). Then for \( n \geq N \)

\[
\left| \frac{n+3}{n^2-3} - 0 \right| = \left| \frac{n+3n}{\frac{1}{2}n^2} \right| = \frac{8}{n} \leq \frac{8}{N} < \epsilon,
\]

so we are done. \( \square \)

**Exercise 4.1.7:** (a) Prove \( \lim_{n \to \infty} \frac{1}{2+3n} = 0. \)

*Proof:* Let \( \epsilon > 0 \) and \( N \geq \frac{1}{3\epsilon} \). Then for \( n \geq N \)

\[
\left| \frac{1}{2+3n} - 0 \right| \leq \frac{1}{3n} \leq \frac{1}{3N} < \epsilon,
\]

so we are done. \( \square \)

(c) Prove \( \lim_{n \to \infty} \frac{6n^2+3n}{2n^2-5} = 3. \)

*Proof:* Let \( \epsilon > 0 \) and \( N > \max\{3, \frac{18}{\epsilon}\} \). Then for \( n \geq N \)

\[
\left| \frac{6n^2+3n}{2n^2-5} - 3 \right| = \left| \frac{6n^2+3n-6n^2+15}{2n^2-5} \right| = \left| \frac{3n+15}{2n^2-5} \right| \leq \frac{18n^2}{2n^2} = \frac{18}{n} \leq \frac{18}{N} < \epsilon,
\]

so we are done. \( \square \)

(d) Show \( \lim_{n \to \infty} \sqrt{n+1} = 0. \)

*Proof:* Let \( \epsilon > 0 \) and \( N > \frac{1}{\epsilon^2} \). Then for \( n \geq N \)

\[
\left| \frac{\sqrt{n}}{n+1} - 0 \right| \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \epsilon,
\]

so we are done. \( \square \)

**Exercise 4.1.8:** (a) Show that \( a_n = 2n \) is divergent.

*Proof:* Note that this sequence is not bounded. To see this, assume by way of contradiction that we have some bound \( M \). Then let \( N > M \) and observe that \( |a_N| = |2N| > M \), contradicting the fact that \( M \) is a bound and confirming that \( a_n \) is indeed unbounded. But then if \( a_n \) is not bounded it cannot be convergent, by Theorem 4.1.13. \( \square \)

**Exercise 4.1.10:** Find an example of each of the following.
(a) A convergent sequence of rational numbers having an irrational limit.
Proof: Define a sequence of partial decimal expansions of $\sqrt{2}$. That is, let

$$a_0 = 1, \quad a_1 = 1.4, \quad a_2 = 1.41, \quad a_3 = 1.414, \ldots$$

Then every $a_n$ will certainly be rational, as each will have a finite decimal expansion. However, $(a_n) \to \sqrt{2}$, which is irrational. □

(b) A convergent sequence of irrational numbers having a rational limit.

Proof: We claim that $(a_n) = \frac{\pi}{10^n}$ is such a sequence. Observe that for each $n$, $a_n$ will simply be a copy of $\pi$ with the decimal moved to a different location. As a result, every $a_n$ will still have a non-terminating, non-repeating decimal expansion, which means it will still be irrational. However, we also have that $(a_n) \to 0$. So $(a_n)$ is indeed a sequence of irrational numbers with a rational limit. □

**Exercise 4.2.4:** (a) Show that if $\lim s_n = s$ and $k \in \mathbb{R}$, then $\lim (ks_n) = ks$ and $\lim (k + s_n) = k + s$.

Proof: We begin by showing that $\lim (ks_n) = ks$. Let $\epsilon > 0$. Since $s_n \to s$, there exists $N$ such that for $n \geq N$ we have $|s_n - s| < \frac{\epsilon}{|k|}$. Then observe for such $n$ that

$$|ks_n - ks| = |k||s_n - s| < |k|\frac{\epsilon}{|k|} = \epsilon,$$

as desired.

We now show that $\lim (k + s_n) = k + s$. Let $\epsilon > 0$. Since $s_n \to s$, there exists $N$ such that for $n \geq N$ we have $|s_n - s| < \epsilon$. Then observe for such $n$ that

$$|(k + s_n) - (k + s)| = |s_n - s| < \epsilon,$$

as desired. □

(b) Show that if $t_n \to t$ and $t_n \geq 0$ for all $n \in \mathbb{N}$, then $t \geq 0$. Proof: Let $(s_n)$ be the constant 0 sequence, then $(s_n) \to 0$ and $s_n \leq t_n$ for all $n$. Then Theorem 4.2.4 tells us that $s \leq t$, so $0 \leq t$. □

**Exercise 4.2.6:** For each of the following, prove or give a counterexample.

(a) If $(s_n)$ and $(t_n)$ are divergent sequences, then $(s_n + t_n)$ diverges.

Counterexample: Let $(s_n) = (-1)^n$ and $(t_n) = (-1)^{n+1}$. Then both diverge but $(s_n) + (t_n) = (-1)^n + (-1)^{n+1} = 0$. So $(s_n + t_n) = 0$, a constant sequence that is obviously convergent. □
(b) If \((s_n)\) and \((t_n)\) are divergent sequences, then \((s_n t_n)\) diverges.

**Counterexample:** Let \((s_n) = (-1)^n\) and \((t_n) = (-1)^{n+1}\). Then both diverge but \((s_n)(t_n) = (-1)^n(-1)^{n+1} = (-1)^{2n+1} = -1\). So \((s_n t_n) = -1\), a constant sequence that is obviously convergent. □

c) If \((s_n)\) and \((s_n + t_n)\) are convergent sequences, then \((t_n)\) converges.

**Proof:** Say that \((s_n) \rightarrow s\) and \((s_n + t_n) \rightarrow s + t\). Then we know from the limit theorems that \((-s_n) \rightarrow -s\) and \((t_n) = (t_n + s_n - s_n) = (s_n + t_n - s_n) \rightarrow s + t - s = t\), as desired. □

d) If \((s_n)\) and \((s_n t_n)\) are convergent sequences, then \((t_n)\) converges.

**Counterexample:** Let \((s_n) = \frac{1}{n}\) and \((t_n) = (-1)^n\). Then \((s_n) \rightarrow 0\) and \((t_n)\) diverges but \((s_n t_n) \rightarrow 0\). □

**Exercise 4.2.9:** Suppose that \((s_n)\) and \((t_n)\) are sequences such that \(s_n \leq t_n\) for all \(n \in \mathbb{N}\).

Prove the following:

(a) If \(\lim s_n = +\infty\), then \(\lim t_n = +\infty\).

**Proof:** Let \(M \in \mathbb{R}\). Then there exists \(N\) such that for all \(n \geq N\) we have \(M < s_n \leq t_n\). Hence \((t_n) \rightarrow +\infty\). □

(b) If \(\lim t_n = -\infty\), then \(\lim s_n = -\infty\).

**Proof:** Let \(M \in \mathbb{R}\). Then there exists \(N\) such that for all \(n \geq N\) we have \(M > t_n \geq s_n\). Hence \((s_n) \rightarrow -\infty\). □

**Exercise 4.2.10:** Show that \(\lim \frac{1}{s_n} = 0\) implies that \(\lim s_n = +\infty\), given that \((s_n)\) is a positive sequence.

**Proof:** Let \(M \in \mathbb{R}^+\) and set \(\epsilon = \frac{1}{M}\). Then there exists \(N\) such that for all \(n \geq N\) we have \(|\frac{1}{s_n} - 0| < \epsilon = \frac{1}{M}\), which implies \(\frac{1}{s_n} < \frac{1}{M}\), so then \(M < s_n\), since everything is positive. Then by definition we have that \((s_n) \rightarrow +\infty\). □

**Exercise 4.3.3:** Prove that each sequence is monotone and bounded. Then find the limit.

(a) \(s_1 = 1\) and \(s_{n+1} = \frac{1}{5}(s_n + 7)\) for \(n \geq 1\).

**Proof:** We first claim that \((s_n)\) is increasing, and we shall show this inductively.

Base case: For \(n = 1\), note that \(s_1 = 1 \leq \frac{8}{5} = \frac{1}{5}(1 + 7) = s_2\).

Inductive step: Assume that \(s_k \leq s_{k+1}\); we want to show that this is true for \(k + 1\). Now
observe that
\[ s_k \leq s_{k+1} \implies \frac{1}{5}(s_k + 7) \leq \frac{1}{5}(s_{k+1} + 7) \implies s_{k+1} \leq s_{k+2}. \]

Hence, the induction holds and we have that \((s_n)\) is increasing, as claimed.

We now claim that \((s_n)\) is bounded above by 2, and we shall also show this inductively.

Base case: For \(n = 1\) we have that \(s_1 = 1 < 2\).

Inductive step: Assume that \(s_k < 2\); we want to show that this is true for \(k+1\). Now observe that
\[ s_{k+1} = \frac{1}{5}(s_k + 7) < \frac{1}{5}(2 + 7) = \frac{9}{5} < 2, \]
so then induction holds, proving our claim that \(s_n < 2\) for all \(n\).

As \((s_n)\) has been shown to be increasing and bounded above, we know that \((s_n)\) is convergent. Its limit will then be given by the solution to the equation
\[ s = \frac{1}{5}(s + 7), \]
which is \(s = \frac{7}{4}\). □

(c) \(s_1 = 2\) and \(s_{n+1} = \frac{1}{4}(2s_n + 7)\) for \(n \geq 1\).

Proof: We first claim that \((s_n)\) is increasing, and we shall show this inductively.

Base case: For \(n = 1\), note that \(s_2 = 1 \leq \frac{11}{4} = \frac{1}{4}(2(2) + 7) = s_2\).

Inductive step: Assume that \(s_k \leq s_{k+1}\); we want to show that this is true for \(k+1\). Now observe that
\[ s_k \leq s_{k+1} \implies \frac{1}{4}(2s_k + 7) \leq \frac{1}{4}(2s_{k+1} + 7) \implies s_{k+1} \leq s_{k+2}. \]

Hence, the induction holds and we have that \((s_n)\) is increasing, as claimed.

We now claim that \((s_n)\) is bounded above by 4, and we shall also show this inductively.

Base case: For \(n = 1\) we have that \(s_1 = 2 < 4\).

Inductive step: Assume that \(s_k < 4\); we want to show that this is true for \(k+1\). Now observe that
\[ s_{k+1} = \frac{1}{4}(2s_k + 7) < \frac{1}{4}(2(4) + 7) = \frac{15}{4} < 4, \]
so then induction holds, proving our claim that \(s_n < 4\) for all \(n\).
As \((s_n)\) has been shown to be increasing and bounded above, we know that \((s_n)\) is convergent. Its limit will then be given by the solution to the equation
\[ s = \frac{1}{4}(2s + 7), \]
which is \(s = \frac{7}{2}. \) □

(e) \(s_1 = 5\) and \(s_{n+1} = \sqrt{4s_n + 1}\) for \(n \geq 1.\)

Proof: We first claim that \((s_n)\) is decreasing, and we shall show this inductively.

Base case: For \(n = 1,\) note that \(s_1 = 5 \geq \sqrt{21} = \sqrt{4(5)+1} = s_2.\)

Inductive step: Assume that \(s_k \geq s_{k+1};\) we want to show that this is true for \(k+1.\) Now observe that
\[ s_k \geq s_{k+1} \implies \sqrt{4s_k + 1} \geq \sqrt{4s_{k+1} + 1} \implies s_{k+1} \geq s_{k+2}. \]

Hence, the induction holds and we have that \((s_n)\) is decreasing, as claimed.

We now claim that \((s_n)\) is bounded below by \(3,\) and we shall also show this inductively.

Base case: For \(n = 1\) we have that \(s_1 = 5 > 3.\)

Inductive step: Assume that \(s_k > 3;\) we want to show that this is true for \(k+1.\) Now observe that
\[ s_{k+1} = \sqrt{4s_k + 1} > \sqrt{4(3) + 1} = \sqrt{13} > 3, \]
so then induction holds, proving our claim that \(s_n > 3\) for all \(n.\)

As \((s_n)\) has been shown to be decreasing and bounded below, we know that \((s_n)\) is convergent. Its limit will then be given by one of the solutions to the equation
\[ s = \sqrt{4s + 1}, \]
so then \(s = 2 \pm \sqrt{3}.\) Since we know that \(s \geq 3,\) it follows that \(s = 2 + \sqrt{3}. \) □

Exercise 4.3.12: (a) Show that if \((s_n)\) is an unbounded increasing sequence, then \(\lim s_n = +\infty.\)

Proof: Note \((s_n)\) increasing means \(s_1 \leq s_n\) for all \(n,\) so \((s_n)\) is bounded below. Hence, \((s_n)\) must be unbounded above. Let \(M \in \mathbb{R}^+.\) Since \((s_n)\) is unbounded above, there exists \(N\) such that \(s_N > M.\) Then since \((s_n)\) is increasing we know \(s_n \geq s_N > M\) for all \(n \geq N.\)

Since \(M\) is arbitrary, this implies that \((s_n) \to \infty. \) □
(b) Show that if \((s_n)\) is an unbounded decreasing sequence, then \(\lim s_n = -\infty\).

Proof: Note \((s_n)\) decreasing means \(s_1 \geq s_n\) for all \(n\), so \((s_n)\) is bounded above. Hence, \((s_n)\) must be unbounded below. Let \(M \in \mathbb{R}^-\). Since \((s_n)\) is unbounded below, there exists \(N\) such that \(s_N < M\). Then since \((s_n)\) is decreasing we know \(s_n \leq s_N < M\) for all \(n \geq N\). Since \(M\) is arbitrary, this implies that \((s_n) \to -\infty\). □

**Exercise 4.4.8:** If \((s_n)\) is a subsequence of \((t_n)\) and \((t_n)\) is a subsequence of \((s_n)\), can we conclude that \((s_n) = (t_n)\)? Prove or give a counterexample.

Counterexample: Let

\[(s_n) = 1, 0, 1, 0, 1, 0, \ldots\]
\[(t_n) = 0, 1, 0, 1, 0, 1, \ldots\]

Then clearly \((s_n) \neq (t_n)\), as \(s_1 \neq t_1\). However, \((s_n) \subseteq (t_n)\) and \((t_n) \subseteq (s_n)\). □
Chapter 5

Exercise 5.1.6: Prove the following limits.
(a) \( \lim_{x \to 3} (x^2 - 5x + 1) = -5. \)

Proof: Observe that
\[
| (x^2 - 5x + 1) - (-5) | = |x^2 - 5x + 6| = |x - 3||x - 2|.
\]
Say \( |x - 3| < 1; \) then we have that
\[
|x - 2| = |x - 3 + 1| \leq |x - 3| + 1 < 1 + 1 = 2.
\]
Now let \( \epsilon > 0. \) Set \( \delta < \min\{1, \frac{\epsilon}{2}\}. \) Then when \( 0 < |x - 3| < \delta \) we have
\[
| (x^2 - 5x + 1) - (-5) | = |x - 3||x - 2| < \delta \cdot 2 < \frac{\epsilon}{2} \cdot 2 = \epsilon.
\]
Since our choice of \( \epsilon \) was arbitrary, we have that \( \lim_{x \to 3} (x^2 - 5x + 1) = -5. \) □

(c) \( \lim_{x \to 2} x^3 = 8. \)

Proof: Observe that
\[
| x^3 - 8 | = |x - 2||x^2 + 2x + 4|.
\]
Say \( |x - 2| < 1; \) then we have that
\[
|x^2 + 2x + 4| \leq |x|^2 + 2|x| + 4
= |x - 2 + 2|^2 + 2|x - 2 + 2| + 4
\leq (|x - 2| + 2)^2 + 2|x - 2| + 4 + 4
= |x - 2|^2 + 6|x - 2| + 12
< 1^2 + 6(1) + 12 = 19.
\]
Now let \( \epsilon > 0. \) Set \( \delta < \min\{1, \frac{\epsilon}{19}\}. \) Then when \( 0 < |x - 2| < \delta \) we have
\[
| x^3 - 8 | = |x - 2||x^2 + 2x + 4| < \delta \cdot 19 < \frac{\epsilon}{19} \cdot 19 = \epsilon.
\]
Since our choice of \( \epsilon \) was arbitrary, we have that \( \lim_{x \to 2} x^3 = 8. \) □

Exercise 5.1.8: Let \( f : D \to \mathbb{R} \) and let \( c \) be an accumulation point of \( D. \) Suppose that \( \lim_{x \to c} f(x) = L. \)

(a) Prove that \( \lim_{x \to c} |f(x)| = |L|. \)

Proof: As we know that \( \lim_{x \to c} f(x) = L, \) we have the following:

“For all \( \epsilon > 0, \) there exists \( \delta > 0 \) such that for \( |x - c| < \delta \) we have \( |f(x) - L| < \epsilon. \)”
Applying the reverse triangle inequality to the final inequality in this statement gives

\[ |f(x)| - |L| \leq |f(x) - L| < \epsilon. \]

So then we can rephrase the above statement to:

“For all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for \( |x - c| < \delta \) we have \( |f(x)| - |L| < \epsilon \).”

But then we may observe that this is exactly the definition of convergence of \( f(x) \) to \( L \) at \( c \), so we know that \( \lim_{x \to c} |f(x)| = |L| \). □

**Exercise 5.1.9:** Determine whether or not the following limits exist. Justify your answers.

(a) \( \lim_{x \to 0^+} \frac{1}{x} \).

*Proof:* Consider the sequence \( \left( \frac{1}{n} \right) \). Then we know that \( \left( \frac{1}{n} \right) \to 0 \) and \( \frac{1}{n} > 0 \) for all \( n \). However, note that \( (\frac{1}{1/n}) = (n) \), which is certainly a divergent sequence. Then Theorem 5.1.10 tells us that \( \frac{1}{x} \) does not have a limit at 0. □

(b) \( \lim_{x \to 0^+} |\sin \frac{1}{x}| \).

*Proof:* Consider the sequence \( \left( \frac{2}{n\pi} \right) \). Then we know that \( \left( \frac{2}{n\pi} \right) \to 0 \) and \( \frac{2}{n\pi} > 0 \) for all \( n \). However, note that

\[
\left( |\sin \frac{2}{n\pi}| \right) = \left( |\sin \frac{n\pi}{2}| \right) = (-1)^n,
\]

which is a divergent sequence. Theorem 5.1.10 then tells us that \( \frac{1}{x} \) does not have a limit at 0. □

(c) \( \lim_{x \to 0^+} x \sin \frac{1}{x} \).

*Proof:* We claim that this limit exists, and \( L = 0 \). Observe that for \( 0 < |x| < 1 \) we have that

\[ |x \sin \frac{1}{x}| = |x| |\sin \frac{1}{x}| < |x| \cdot 1 = |x| \]

Then let \( \epsilon > 0 \) and set \( \delta < \min\{1, \epsilon\} \). Then for \( |x| < \delta \) we have that

\[ |x \sin \frac{1}{x}| = |x| |\sin \frac{1}{x}| < |x| < \delta < \epsilon, \]

which proves our claim. □

**Exercise 5.1.11:** Let \( f : D \to \mathbb{R} \) and let \( c \) be an accumulation point of \( D \). Show that the following are equivalent.

(a) \( f \) does not have a limit at \( c \).
There exists a sequence \((s_n)\) in \(D\) with each \(s_n \neq c\) such that \((s_n)\) converges to \(c\), but \((f(s_n))\) is not convergent in \(\mathbb{R}\).

**Proof:** 
((a)⇒(b)) We will show this contrapositively, so assume that (b) is false. Then any sequence \((s_n)\) in \(D\) converging to \(c\) with \(s_n \neq c\) must have the property that \((f(s_n))\) is convergent in \(\mathbb{R}\). We want to show that all such sequences converge to the same value.

Consider two sequences \((s_n)\) and \((t_n)\) in \(D\) with \(s_n, t_n \neq c\) and \((s_n) \to c\) and \((t_n) \to c\). Let \(L_s\) and \(L_t\) be the limits of \((f(s_n))\) and \((f(t_n))\) respectively. Now consider the sequence \((u_n)\) defined by

\[
(u_n) = s_1, t_1, s_2, t_2, s_3, t_3, \ldots
\]

Then \((u_n) \to c\) and \((f(u_n))\) has some limit, call it \(L_u\). But as \((s_n)\) and \((t_n)\) are subsequences of \((u_n)\), we know that \((f(s_n))\) and \((f(t_n))\) are subsequences of \((f(u_n))\). But any subsequence of a convergent sequence must have the same limit, which tells us that \(L_s = L_u\) and \(L_t = L_u\), so \(L_s = L_t\), as desired.

Hence, we know that any two sequences converging to \(c\) (but never equaling \(c\)) must have same value \(L\) as the limit of their images, so any sequence converging to \(c\) (and not equaling \(c\)) must have the limit \(L\) in its image. Then converse direction of Theorem 5.1.8 implies \(\lim_{x \to c} f(x) = L\). So then \(f\) does have a limit at \(c\), which is the negation of (a). This completes our proof of the contrapositive, so we have that (a) implies (b).

((b)⇒(a)) Now assume that (b) is true. Then we know that there exists some sequence \((s_n) \to c\) with \(s_n \neq c\) such that \((f(s_n)) \not\to L\) for any choice of \(L\). Observe that this is the negation of the second half of Theorem 5.1.8 for any choice of \(L\), so then the first half of Theorem 5.1.8 is also false for any choice of \(L\); specifically, the limit of \(f\) at \(c\) does not equal \(L\) for any choice of \(L\). Thus, we may conclude that the limit of \(f\) at \(c\) cannot be any \(L\), which is equivalent to saying that \(f\) does not have a limit at \(c\). As this is exactly (a), we are finished.

**Exercise 5.2.6:** Prove or give counterexamples for the following.

(a) If \(f\) is continuous on \(D\) and \(k \in \mathbb{R}\), then \(kf\) is continuous on \(D\).

**Proof:** Let \(\epsilon > 0\) and \(c \in D\). Since \(f\) is continuous at \(c\), we know that there exists \(\delta > 0\) such that \(|f(x) - f(c)| < \frac{\epsilon}{|k|}\) when \(|x - c| < \delta\) with \(x \in D\).

Then for \(x \in D\) with \(|x - c| < \delta\) we have

\[
|kf(x) - kf(c)| = |k||f(x) - f(c)| < |k|\frac{\epsilon}{|k|} = \epsilon.
\]

So then \(kf(x)\) is indeed continuous at \(c\). Since \(c\) is arbitrary, \(kf(x)\) is continuous in \(D\).
(b) If \( f \) and \( f + g \) are continuous on \( D \), then \( g \) is continuous on \( D \).

**Proof:** Let \( \epsilon > 0 \) and \( c \in D \). Then since \( f \) and \( f + g \) are continuous at \( c \), there exists \( \delta_1 > 0 \) such that

\[
|f(x) - f(c)| < \frac{\epsilon}{2} \quad \text{when } |x - c| < \delta_1 \quad \text{with } x \in D,
\]

and there exists \( \delta_2 > 0 \) such that

\[
|(f + g)(x) - (f + g)(c)| < \frac{\epsilon}{2} \quad \text{when } |x - c| < \delta_2 \quad \text{with } x \in D,
\]

Now let \( \delta = \min\{\delta_1, \delta_2\} \) and observe for \( |x - c| < \delta \) with \( x \in D \) that

\[
|g(x) - g(c)| = |g(x) + f(x) - f(x) - g(c) + f(c) - f(c)|
\]

\[
= |((f + g)(x) - (f + g)(c)) - (f(x) - f(c))| \\
\leq |(f + g)(x) - (f + g)(c)| + |f(x) - f(c)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Hence, we have that \( g(x) \) is continuous at all \( c \in D \).

(c) If \( f \) and \( fg \) are continuous on \( D \), then \( g \) is continuous on \( D \).

**Counterexample:** Let \( f(x) = 0 \) and

\[
g(x) = \begin{cases} 
1 & x \geq 0 \\
-1 & x < 0
\end{cases}
\]

Then \((f \cdot g)(x) = 0\), which is obviously continuous, but \( g \) is not continuous.

(d) If \( f^2 \) is continuous on \( D \), then \( f \) is continuous on \( D \).

**Counterexample:** Let

\[
f(x) = \begin{cases} 
1 & x \geq 0 \\
-1 & x < 0
\end{cases}
\]

Then \( f^2(x) = 1 \), which is obviously continuous, but \( f \) is not continuous.

(e) If \( f \) is continuous on \( D \) and \( D \) is bounded, then \( f(D) \) is bounded.

**Counterexample:** Let \( f(x) = \frac{1}{x} \) and \( D(0,1) \). Then \( f(x) \) is continuous on \( D \) and \( D \) is bounded, but \( f((0,1)) = (1, \infty) \).

(f) If \( f \) and \( g \) are not continuous on \( D \), then \( f + g \) is not continuous on \( D \).

**Counterexample:** Let

\[
f(x) = \begin{cases} 
1 & x \geq 0 \\
-1 & x < 0
\end{cases} \quad \text{and} \quad g(x) = \begin{cases} 
-1 & x \geq 0 \\
1 & x < 0
\end{cases}
\]
Then $f$ and $g$ are both not continuous, but $(f + g)(x) = 0$, which is obviously continuous. □

(g) If $f$ and $g$ are not continuous on $D$, then $fg$ is not continuous on $D$.

Counterexample: Let $f$ and $g$ be as in part (f). Then $f$ and $g$ are not continuous, but $(f \cdot g)(x) = -1$, which is obviously continuous. □

(h) If $f : D \rightarrow E$ and $g : E \rightarrow F$ are not continuous on $D$ and $E$, respectively, then $g \circ f : D \rightarrow F$ is not continuous on $D$.

Counterexample: Define $f, g : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \lfloor x \rfloor$ and

$$
g(x) = \begin{cases} 
1 & x \in \mathbb{Z} \\
0 & x \notin \mathbb{Z}.
\end{cases}
$$

Then both $f$ and $g$ are not continuous, but $g \circ f(x) = 1$, since the output of $f$ is always an integer. □

Exercise 5.2.10: (a) Let $f : D \rightarrow \mathbb{R}$ and define $|f| : D \rightarrow \mathbb{R}$ by $|f|(x) = |f(x)|$. Suppose that $f$ is continuous at $c \in D$. Prove that $|f|$ is continuous at $c$.

Proof: Let $\epsilon > 0$ and $c \in D$. Then there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ when $|x - c| < \delta$ with $x \in D$. Then the reverse triangle inequality gives

$$||f(x)| - |f(c)|| \leq |f(x) - f(c)| < \epsilon$$

when $|x - c| < \delta$ with $x \in D$, so $|f|$ is continuous. □

(b) If $|f|$ is continuous at $c$, does it follow that $f$ is continuous at $c$? Justify your answer.

Counterexample: Let

$$f(x) = \begin{cases} 
1 & x \geq 0 \\
-1 & x < 0.
\end{cases}
$$

Then $f$ is not continuous but $|f(x)| = 1$, which is obviously continuous. □

Exercise 5.2.14: Let $f : D \rightarrow \mathbb{R}$ be continuous at $c \in D$. Prove that there exists an $M > 0$ and a neighborhood $U$ of $c$ such that $|f(x)| \leq M$ for all $x \in U \cap D$.

Proof: Let $\epsilon > 0$. Since $f$ is continuous, there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ when $|x - c| < \delta$ and $x \in D$. Note that

$$|x - c| < \delta \iff x \in (c - \delta, c + \delta).$$
Then by reverse triangle inequality
\[ |f(x) - f(c)| < \epsilon \implies |f(x)| - |f(c)| < \epsilon \implies |f(x)| < \epsilon + |f(c)| \]
when \( x \in (c - \delta, c + \delta) \). Now simply set \( U = (c - \delta, c + \delta) \) and \( M = |f(c)| + \epsilon \) to observe that we have the desired result. \( \square \)

**Exercise 5.2.18:** Suppose that \( f : (a, b) \to \mathbb{R} \) is continuous and that \( f(r) = 0 \) for every rational number \( r \in (a, b) \). Prove that \( f(x) = 0 \) for all \( x \in (a, b) \).

*Proof:* By way of contradiction suppose that \( f(c) > 0 \) for some \( c \). Select \( \epsilon \) such that \( 0 < \epsilon < f(c) \). Then since \( f \) is continuous we know that there exists \( \delta > 0 \) such that \( |f(x) - f(c)| < \epsilon \) when \( |x - c| < \delta \) and \( x \in D \).

Now recall that \( |x - c| < \delta \) if and only if \( x \in (c - \delta, c + \delta) \). Note that this interval will always contain a rational number, regardless of our choice of \( \delta \). Then if \( x_0 \) is rational we see
\[ |f(x_0) - f(c)| = |0 - f(c)| = f(c) < \epsilon, \]
but this contradicts our choice of \( \epsilon \). Hence, \( f(c) = 0 \) for all \( c \in \mathbb{R} \). \( \square \)

**Exercise 5.3.5:** Show that \( 5^x = x^4 \) has at least one real solution.

*Proof:* Define \( f(x) = 5^x - x^4 \), and note that \( f \) is continuous. Further, observe that the equation in the problem statement has a solution exactly for the \( x \) which give \( f(x) = 0 \). Note that \( f(0) = 1 \) and \( f(-1) = \frac{-4}{5} \). Then since \( \frac{-4}{5} < 0 < 1 \), the intermediate value theorem tells us that there exists some \( c \in (-1, 0) \) such that \( f(c) = 0 \). Hence, this \( c \) will be a solution to \( 5^x = x^4 \). \( \square \)

**Exercise 5.3.14:** Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = \sin(1/x) \) if \( x \neq 0 \) and \( f(0) = 0 \).

(a) Show that \( f \) is not continuous at 0.

*Proof:* Let \( \epsilon = \frac{1}{2} \) and observe that for any \( \delta > 0 \) there exists \( x \in (-\delta, \delta) \) such that \( \sin(\frac{1}{x}) = 1 \). Then \( |f(x) - 0| = |1 - 0| = 1 < \frac{1}{2} \). Hence, \( f \) cannot be continuous at 0. \( \square \)

(b) Show that \( f \) has the intermediate value property on \( \mathbb{R} \).

*Proof:* Let \( f(a), f(b) \in f(\mathbb{R}) = [-1, 1] \), and without loss of generality say that \( f(a) \leq f(b) \) and let \( k \) be between them. We have three cases.

Case 1: \( a, b < 0 \).

Since \( f(x) \) is continuous on \( (-\infty, 0) \), we are immediately done because the intermediate value theorem applies.
Case 2: $a, b > 0$.
Since $f(x)$ is continuous on $(0, \infty)$, we are immediately done because the intermediate value theorem applies.

Case 3: $0$ is between $a$ and $b$. (Note that we do not allow $a = b$, as the intermediate value property requires $a \neq b$.)
Since $a$ and $b$ cannot both be $0$, without loss of generality say that $a$ is not. Then we without loss of generality also say that $a < 0$. Then consider $(a, 0)$ and note that $\sin\left(\frac{1}{x}\right)$ oscillates infinitely over this interval, as it is adjacent to $0$. Then $f((a, 0)) = [-1, 1]$, so there exists $c \in (a, 0)$ such that $f(c) = k$ (since $k \in [-1, 1]$). So then $c$ must be between $a$ and $b$, so the intermediate value property holds. □

**Exercise 5.4.5:** Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

**Proof:** Let $\epsilon > 0$.

Note that $f$ is uniformly continuous on $[0, 10]$, since $f$ is continuous and this set is compact.
Then for our $\epsilon$ we have some corresponding $\delta_1 > 0$ such that $|f(x) - f(y)| < \epsilon$ when $|x - y| < \delta_1$ and $x, y \in [0, 10]$.

We want to show that $f$ is uniformly continuous on $[9, \infty)$. First observe that

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = |\sqrt{x} - \sqrt{y}| \left|\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}}.$$

Since we are only considering $x, y \in [9, \infty)$, we know that

$$|f(x) - f(y)| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x - y|}{\sqrt{9} + \sqrt{9}} = \frac{|x - y|}{6}.$$

Now set $\delta_2 < 6\epsilon$. Then for $|x - y| < \delta_2$ and $x, y \in [9, \infty]$ we have

$$|f(x) - f(y)| \leq \frac{|x - y|}{6} < \frac{\delta_2}{6} < \frac{6\epsilon}{6} = \epsilon.$$

Hence, we have that $f$ is uniformly continuous on $[9, \infty)$.

Now set $\delta = \min\{1, \delta_1, \delta_2\}$. Then for $|x - y| < \delta$ we have that $x, y \in [0, 10]$ or $x, y \in [9, \infty)$.
Hence, we may consider two cases.

Case 1: $x, y \in [0, 10]$.
Then for our given $\epsilon$, we have that $|x - y| < \delta \leq \delta_1$, so $|f(x) - f(y)| < \epsilon$.

Case 2: $x, y \in [9, \infty)$.
Then for our given $\epsilon$, we have that $|x - y| < \delta \leq \delta_2$, so $|f(x) - f(y)| < \epsilon$. 

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Thus, we have that $f$ is uniformly continuous on $[0, \infty)$. □

**Exercise 5.4.7:** Let $f : D \rightarrow \mathbb{R}$ be uniformly continuous on $D$ and let $k \in \mathbb{R}$. Prove that the function $kf$ is uniformly continuous on $D$.

**Proof:** We have two cases.

Case 1: $k = 0$.
If $k = 0$ then $kf(x) = 0$, which is trivially uniformly continuous.

Case 2: $k \neq 0$.
Let $\epsilon > 0$. Then there exists some corresponding $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{|k|}$ when $|x - y| < \delta$ and $x, y \in D$, since $f$ is uniformly continuous. Then for such $x, y$ we have

$$|kf(x) - kf(y)| = |k||f(x) - f(y)| < |k|\frac{\epsilon}{|k|} = \epsilon.$$  

Hence, $kf$ is uniformly continuous. □
Chapter 6

Exercise 6.1.10: Suppose $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ are differentiable at $c \in I$.

(a) Prove if $k \in \mathbb{R}$, then the function $kf$ is differentiable at $c$ and $(kf)'(c) = k \cdot f'(c)$.

Proof: Observe

$$(kf)'(c) = \lim_{x \to c} \frac{kf(x) - kf(c)}{x - c} = k \left( \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \right) = k \cdot f'(c).$$

This completes the proof. □

(b) Prove $f + g$ is differentiable at $c$ and $(f + g)'(c) = f'(c) + g'(c)$.

Proof: Observe that

$$(f + g)'(c) = \lim_{x \to c} \frac{f(x) + g(x) - f(c) - g(c)}{x - c}
= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c}
= f'(c) + g'(c).$$

This completes the proof. □

Exercise 6.1.13: Let $f$, $g$, and $h$ be real-valued functions that are differentiable on an interval $I$. Prove that the product function $fgh : I \rightarrow \mathbb{R}$ is differentiable on $I$ and find $(fgh)'$.

Proof: Set $k(x) = (g \cdot h)(x)$. The product rules gives that

$$k'(x) = g(x)h'(x) + h(x)g'(x).$$

We can combine this fact with another use of the product rule again to see the following:

$$(f \cdot g \cdot h)'(x) = (f \cdot k)'(x)
= f(x)k'(x) + k(x)f'(x)
= f(x)(g(x)h'(x) + h(x)g'(x)) + (g \cdot h)(x)f'(x)
= f(x)g(x)h'(x) + f(x)h(x)g'(x) + g(x)h(x)f'(x).$$

This is then the desired derivative. □

Exercise 6.2.5: Use the mean value theorem to establish the following inequalities.

(a) $e^x > 1 + x$ for $x > 0$.

Proof: The Mean Value Theorem implies that there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}.$$
Substituting $f(x) = e^x$ and recalling $f'(x) = e^x$ then gives

$$e^c = \frac{e^x - e^0}{x - 0} = \frac{e^x - 1}{x}.$$ 

As $e^x$ is increasing for all $x$, we know that $e^0 < e^c$ for all $c \in (0, x)$. Then

$$1 = e^0 < e^c = \frac{e^x - 1}{x} \implies x + 1 < e^x$$

for $x > 0$, as desired. □

(f) $\sin x \leq x$ for $x \geq 0$.

**Proof:** The Mean Value Theorem implies that there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}.$$ 

Substituting $f(x) = \sin x$ and recalling $f'(x) = \cos x$ then gives

$$\cos c = \frac{\sin x - \sin 0}{x - 0} = \frac{\sin x}{x}.$$ 

Since the supremum of $\cos x$ on $(0, x)$ will be 1, we have

$$1 \geq \cos c = \frac{\sin x}{x} \implies x \geq \sin x$$

for $x \geq 0$, as desired. □

(j) $|\frac{\sin ax - \sin bx}{x}| \leq |a - b|$ for $x \neq 0$.

**Proof:** Let $x \neq 0$ and $a < b$. Set $f(x) = \sin x$. Then the Mean Value Theorem implies that there exists $c$ such that

$$\cos c = \frac{\sin(ax) - \sin(bx)}{ax - bx}.$$ 

$$\implies |\cos c| = \left| \frac{\sin(ax) - \sin(bx)}{ax - bx} \right|$$

$$\implies 1 \geq \frac{1}{|a - b|} \left| \frac{\sin(ax) - \sin(bx)}{x} \right|$$

$$\implies |a - b| \geq \left| \frac{\sin(ax) - \sin(bx)}{x} \right|.$$ 

As the final line is the desired result, we are done. □

**Exercise 6.2.8:** Suppose that $f$ is differentiable on an interval $I$. Prove the following:

(a) $f$ is increasing on $I$ iff $f'(x) \geq 0$ for all $x \in I$. 

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Proof: \((\Rightarrow)\) Assume that \(f\) is increasing and fix \(x_0 \in I\). Then since we know \(f\) is differentiable on \(I\),
\[
f'(x) = \lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x} = \lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x}.
\]
Note that the right quotient is never negative, as \(f(x_0) \geq f(x)\) since \(x_0 \geq x\) and \(f\) is increasing. Then the limit of nonnegative values must also be nonnegative, so \(f'(x) \geq 0\).

\((\Leftarrow)\) Assume \(f'(x) \geq 0\). Let \(a, b \in I\) with \(a < b\). Then the Mean Value Theorem implies there exists \(c \in I\) such that
\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]
Since we know \(f'(c) \geq 0\) and \(b - a \geq 0\), it follows that \(f(b) - f(a) \geq 0\), so \(f\) is increasing. \(\square\)

(b) \(f\) is decreasing on \(I\) iff \(f'(x) \leq 0\) for all \(x \in I\).

Proof: \((\Rightarrow)\) Assume that \(f\) is decreasing and fix \(x_0 \in I\). Then since we know \(f\) is differentiable on \(I\),
\[
f'(x) = \lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x} = \lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x}.
\]
Note that the right quotient is never positive, as \(f(x_0) \leq f(x)\) since \(x_0 \geq x\) and \(f\) is decreasing. Then the limit of nonpositive values must also be nonpositive, so \(f'(x) \leq 0\).

\((\Leftarrow)\) Assume \(f'(x) \leq 0\). Let \(a, b \in I\) with \(a < b\). Then the Mean Value Theorem implies there exists \(c \in I\) such that
\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]
Since we know \(f'(c) \leq 0\) and \(b - a \geq 0\), it follows that \(f(b) - f(a) \leq 0\), so \(f\) is decreasing. \(\square\)

Exercise 6.2.12: Let \(f\) be differentiable on \(\mathbb{R}\). Suppose that \(f(0) = 0\) and that \(1 \leq f'(x) \leq 2\) for all \(x \geq 0\). Prove that \(x \leq f(x) \leq 2x\) for all \(x \geq 0\).

Proof: We have two cases.

Case 1: \(x > 0\).

If \(x > 0\), the Mean Value Theorem there exists \(c \in (0, x)\) such that
\[
f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.
\]
Combining this with the given information then tells us that
\[
1 \leq f'(c) \leq 2 \implies 1 \leq \frac{f(x)}{x} \leq 2 \implies x \leq f(x) \leq 2x
\]
for \(x > 0\).
Case 2: $x = 0$.
If $x = 0$, then
\[ x \leq f(x) \leq 2x \] becomes $0 \leq 0 \leq 0$,
so the desired inequality holds in this case as well.

Hence, we may conclude that
\[ x \leq f(x) \leq 2x \quad \forall x \geq 0, \]
as desired. $\square$

Exercise 6.3.8: Let $f : (a, \infty) \to \mathbb{R}$. Prove: If the limit of $f$ as $x \to \infty$ exists, then it is unique.

Proof: Let $L = \lim_{x \to \infty} f(x)$; then we have have that for all $\epsilon > 0$ there exists $N \in \mathbb{R}$ such that for all $x > N$ we have $|f(x) - L| < \epsilon$. Now say that $f(x)$ has another limit at infinity, $L'$, and by way of contradiction say $L' \neq L$. Thne $|L - L'| = \delta > 0$. Pick $\epsilon' = \frac{\delta}{2}$ and choose $N$ such that for $x > N$ we have $|f(x) - L| < \frac{\delta}{2}$. Then for $x > N$ we have via the reverse triangle inequality that
\[ |f(x) - L'| = |f(x) - L + L - L'| \geq |L - L'| - |f(x) - L| \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}. \]
But then $|f(x) - L'|$ can never be less than $\frac{\delta}{2}$, which contradicts that $\lim_{x \to \infty} f(x) = L'$. Hence $L$ must be the unique limit at $\infty$. $\square$

Exercise 6.3.10: Let $f : (a, \infty) \to \mathbb{R}$ and let $k \in \mathbb{R}$. Prove that $\lim_{x \to \infty} k/f(x) = 0$ whenever $\lim_{x \to \infty} f(x) = \infty$.

Proof: From the limit laws, we know that
\[ \lim_{x \to \infty} \frac{k}{f(x)} = k \lim_{x \to \infty} \frac{1}{f(x)}. \]
Let $\epsilon > 0$. Then we can write $\epsilon$ as $\epsilon = 1/\alpha$ for some $\alpha \in \mathbb{R}^+$. Since $\lim_{x \to \infty} f(x) = \infty$, there exists $N \in \mathbb{R}$ such that for all $x > N$ we have $f(x) > \infty$. Then for $x > N$ we have
\[ \left| \frac{1}{f(x)} - 0 \right| < \left| \frac{1}{\alpha} \right| = \epsilon. \]
Hence, $\lim_{x \to \infty} 1/f(x) = 0$, so we may conclude that
\[ \lim_{x \to \infty} \frac{k}{f(x)} = 0, \]
as desired. □

**Exercise 6.4.6:** Show that if \( x \in [0, 1] \), then

\[
x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \leq \ln(1 + x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}.
\]

**Proof:** To show this, we will calculate the fourth-degree Taylor polynomial for \( \ln(1 + x) \) at \( c = 0 \). First observe that

\[
\begin{align*}
f(x) &= \ln(1 + x) \quad & f(0) &= 0, \\
\quad f'(x) &= \frac{1}{1 + x} \quad & f'(0) &= 1, \\
\quad f''(x) &= -\frac{1}{(1 + x)^2} \quad & f''(0) &= -1, \\
\quad f'''(x) &= \frac{2}{(1 + x)^3} \quad & f'''(0) &= 2, \quad \text{and} \\
\quad f^{(4)}(x) &= -\frac{6}{(1 + x)^4} \quad & f^{(4)}(0) &= -6.
\end{align*}
\]

Then we have that

\[
p_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.
\]

Hence,

\[
x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \leq \ln(1 + x) \leq x - \frac{x^2}{2} + \frac{x^3}{3},
\]

as desired. □

**Exercise 6.4.9:** Suppose that \( f \) is defined in a neighborhood of \( c \) and suppose that \( f''(c) \) exists.

(a) Show that

\[
f''(c) = \lim_{h \to 0} \frac{f(c + h) + f(c - h) - 2f(c)}{h^2}.
\]

**Proof:** Notice that the direct substitution of \( h = 0 \) gives the indeterminant form \( \frac{0}{0} \). Then we may apply L’Hopital’s rule, we yields

\[
\lim_{h \to 0} \frac{f(c + h) + f(c - h) - 2f(c)}{h^2} = \lim_{h \to 0} \frac{f'(c + h) - f'(c - h)}{2h}.
\]

Recall Exercise 6.1.17(b), which states that

\[
f'(x) = \lim_{h \to 0} \frac{f(c + h) - f(c - h)}{2h}.
\]
So then replacing $f$ with $f'$ gives that

$$\lim_{h \to 0} \frac{f'(c + h) - f'(c - h)}{2h} = f''(x),$$

which completes our proof. □

(b) Give an example where the limit in part (a) exists but $f''(c)$ does not exist. *Proof:*

Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}.$$

Then we know that the second derivative does not exist at $x = 0$, since

$$f''(x) = \begin{cases} 2 & x > 0 \\ -2 & x < 0 \end{cases}.$$

But if we consider the limit from part (a) and set $c = 0$ we see that

$$\lim_{h \to 0} \frac{f(0 + h) + f(0 - h) - 2f(0)}{h^2} = \lim_{h \to 0} \frac{h^2 - (-h)^2 - 2(0)}{h^2} = \lim_{h \to 0} \frac{0}{h^2} = 0,$$

so the limit does indeed exist. □
Chapter 7

Exercise 7.1.6: Suppose that \( f(x) = c \) for all \( x \in [a,b] \). Show that \( f \) is integrable and that \( \int_a^b f(x) \, dx = c(b-a) \).

Proof: We want to show that \( U(f,P) = L(f,P) = c(b-a) \) for all choices of a partition \( P \).
Let \( P \) be a partition of \([a,b]\); then \( P = \{x_0, \ldots, x_n\} \) where \( a = x_0 < \cdots < x_n = b \). Then

\[
U(f,P) = \sum_{i=1}^n \left( \sup_{x \in [x_{i-1}, x_i]} \{ f(x) \} \right) (x_i - x_{i-1}) = \sum_{i=1}^n c(x_i - x_{i-1}) = c(b-a),
\]

and

\[
L(f,P) = \sum_{i=1}^n \left( \inf_{x \in [x_{i-1}, x_i]} \{ f(x) \} \right) (x_i - x_{i-1}) = \sum_{i=1}^n c(x_i - x_{i-1}) = c(b-a).
\]

Hence, \( U(f,P) = L(f,P) = c(b-a) \) for all possible partitions \( P \), so then \( U(f) = L(f) = c(b-a) \), which implies that \( \int_a^b f(x) \, dx = c(b-a) \). □

Exercise 7.1.8: Give an example of a function \( f : [0,1] \to \mathbb{R} \) that is not integrable on \([0,1]\), but \( f^2 \) is integrable on \([0,1]\).

Proof: Let

\[
f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}.
\]

Such an \( f \) will not be integrable, as it is similar to Example 7.1.8. However, \( f^2 \) will indeed be integrable, as it is a constant function (which is integrable according to Exercise 7.1.6). □

Exercise 7.1.14: Let \( f \) and \( g \) be bounded functions on \([a,b]\).

(a) Prove that \( U(f+g) \leq U(f) + U(g) \).

Proof: Let \( \mathcal{P} \) represent the set of partitions of \([a,b]\). Then observe the following:

\[
U(f+g) = \inf_{P \in \mathcal{P}} \{ U(f+g, P) \}
\]

\[
= \inf_{P \in \mathcal{P}} \left\{ \sum_{i=1}^n \left( \sup_{x \in [x_{i-1}, x_i]} \{ (f+g)(x) \} \right) \Delta x_i \right\}
\]

\[
\leq \inf_{P \in \mathcal{P}} \left\{ \sum_{i=1}^n \left( \sup_{x \in [x_{i-1}, x_i]} \{ f(x) \} + \sup_{x \in [x_{i-1}, x_i]} \{ g(x) \} \right) \Delta x_i \right\}
\]

\[
= \inf_{P \in \mathcal{P}} \left\{ \sum_{i=1}^n \left( \sup_{x \in [x_{i-1}, x_i]} \{ f(x) \} \right) \Delta x_i + \sum_{i=1}^n \left( \sup_{x \in [x_{i-1}, x_i]} \{ g(x) \} \right) \Delta x_i \right\}
\]

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\[
\inf_{P \in \mathcal{P}} \{U(f, P) + U(g, P)\} = \inf_{Q \in \mathcal{P}} \{U(f, Q)\} + \inf_{R \in \mathcal{P}} \{U(g, R)\} = U(f) + U(g).
\]

Hence, \(U(f + g) \leq U(f) + U(g)\). However, it still remains to justify steps (1) and (2).

We show (1) first. Let \(f_1, f_2\) be functions on some interval \([s, t]\). Then
\[
\sup_{x \in [s, t]} \{f_1(x) + f_2(x)\} \leq \sup_{x \in [s, t]} \{f_1(x)\} + \sup_{x \in [s, t]} \{f_2(x)\},
\]
as there is more freedom to choose \(x\) on the right side.

Now we show (2). This will be done by showing an inequality in both directions. It follows from the definition of \(\inf\) that
\[
\inf_{P \in \mathcal{P}} \{U(f, P) + U(g, P)\} \geq \inf_{Q \in \mathcal{P}} \{U(f, Q)\} + \inf_{R \in \mathcal{P}} \{U(g, R)\}.
\]
This is similar to the proof of (1); the inequality comes from the additional freedom on the right side.

Now say that \(Q\) and \(R\) are partitions of \([a, b]\); then \(P = Q \cup R\) will be a refinement of both. Theorem 7.1.4 tells us that
\[
U(f, Q) + U(g, R) \geq U(f, P) + U(g, P).
\]
Define
\[
\mathcal{P}' = \{Q \cup R \mid Q, R \in \mathcal{P}\},
\]
and note that \(\mathcal{P}' \subseteq \mathcal{P}\), as every \(P \in \mathcal{P}'\) will certainly be a partition of \([a, b]\). Then
\[
\inf_{P \in \mathcal{P}} \{U(f, P) + U(g, P)\} \leq \inf_{P \in \mathcal{P}'} \{U(f, P) + U(g, P)\} \leq \inf_{Q \in \mathcal{P}} \{U(f, Q)\} + \inf_{R \in \mathcal{P}} \{U(g, R)\}.
\]
Thus, we may conclude that
\[
\inf_{P \in \mathcal{P}} \{U(f, P) + U(g, P)\} = \inf_{Q \in \mathcal{P}} \{U(f, Q)\} + \inf_{R \in \mathcal{P}} \{U(g, R)\},
\]
which proves (2). This completes the proof. \(\square\)

(b) Find an example to show that a strict inequality may hold in part (a).

**Proof:** Define \(f, g : [a, b] \to \mathbb{R}\) via
\[
f(x) = \begin{cases} 
1 & x \in \mathbb{Q} \\
0 & x \notin \mathbb{Q}
\end{cases}
\]
and
\[
g(x) = \begin{cases} 
0 & x \in \mathbb{Q} \\
1 & x \notin \mathbb{Q}
\end{cases}
\]
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Then \((f+g)(x) = 1\), but \(U(f) = 1\), \(U(g) = 1\), and \(U(f+g) = 1\). As \(1 + 1 > 1\), the inequality can certainly be strict. □

**Exercise 7.1.14:** Prove that if \(f\) is decreasing on \([a, b]\), then \(f\) is integrable on \([a, b]\).

**Proof:** As \(f\) is decreasing, \(f(a) \geq f(x) \geq f(b)\), so \(f\) is bounded on \([a, b]\). Now given \(\epsilon > 0\), there exists \(k > 0\) such that \(k[f(a) - f(b)] < \epsilon\).

Let \(P = \{x_0, \ldots, x_n\}\) be a partition of \([a, b]\) satisfying \(\Delta x_i \leq k\) for all \(i\). Since \(f\) is decreasing, we have that \(m_i = f(x_i)\) and \(M_i = f(x_{i-1})\). Hence

\[
U(f, P) - L(f, P) = \sum_{i=1}^{n} [f(x_{i-1}) - f(x_i)] \Delta x_i \\
\leq \sum_{i=1}^{n} [f(x_{i-1}) - f(x_i)] k \\
= k[f(b) - f(a)] < \epsilon.
\]

So then we have that \(f\) is integrable according to Theorem 7.1.9. □

**Exercise 7.2.6:** Let \(f\) be continuous on \([a, b]\) and suppose that, for every integrable function \(g\) defined on \([a, b]\), \(\int_{a}^{b} fg = 0\). Prove that \(f(x) = 0\) for all \(x \in [a, b]\).

**Proof:** By way of contradiction, suppose that \(f\) takes on some nonzero value \(y_0\). Then there exists \(\epsilon > 0\) such that \((y_0 - \epsilon, y_0 + \epsilon)\) does not contain 0. (Without loss of generality let us say that the interval contains all positive numbers.)

Since \(f\) is continuous, \(f^{-1}((y_0 - \epsilon, y_0 + \epsilon))\) will be an open set. Pick \(x_0 \in f^{-1}((y_0 - \epsilon, y_0 + \epsilon))\); since this set is open there exists \(\delta > 0\) such that \((x_0 - \delta, x_0 + \delta) \subseteq f^{-1}((y_0 - \epsilon, y_0 + \epsilon))\). Then it follows that \(f(x) > 0\) for all \(x \in (x_0 - \delta, x_0 + \delta)\). Now say that \(g_0\) is defined by

\[
g_0(x) = \begin{cases} 
1 & x \in (x_0 - \delta, x_0 + \delta) \\
0 & x \not\in (x_0 - \delta, x_0 + \delta) 
\end{cases}
\]

Then \((f \cdot g_0)(x)\) is positive on \((x_0 - \delta, x_0 + \delta)\) and zero elsewhere. But then

\[
\int_{a}^{b} (f \cdot g_0)(x) \, dx > 0,
\]

which is a contradiction. Hence, our assumption that \(f\) was not totally zero must be false, so \(f(x) = 0\) for all \(x \in [a, b]\). □
**Exercise 7.2.9:** Let \( f \) and \( g \) be integrable on \([a, b]\).

(a) Show that their product \( fg \) is integrable on \([a, b]\).

*Proof:* Recall the identity
\[
f \cdot g = \frac{1}{4} [(f + g)^2 - (f - g)^2].
\]

Note that Theorem 7.2.4 tells us that sums, differences, and scalar multiples of integrable functions will be integrable. Practice 7.2.9 tells us that any positive integer power of an integrable function will be integrable. As the above equation allows us represent \( f \cdot g \) using sums, differences, scalar multiples, and positive integer powers of integrable functions, we then have that \( f \cdot g \) is integrable. □

(b) Show that \( \max(f, g) \) and \( \min(f, g) \) are integrable on \([a, b]\).

*Proof:* Recall from Exercise 5.2.11 that
\[
\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g| \quad \text{and} \quad \min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|.
\]

Recall the results stated in part (a) and recall that Corollary 7.2.8 states that the absolute value of an integrable function will be integrable. Then the combination of all these results gives that \( \max(f, g) \) and \( \min(f, g) \) will be integrable, as they are represented above as sums, differences, absolute values, and scalar multiples of integrable functions. □

**Exercise 7.2.10:** Find an example of a function \( f : [0, 1] \rightarrow \mathbb{R} \) such that \( f \) is not integrable on \([0, 1]\), but \( |f| \) is integrable on \([0, 1]\).

*Proof:* Define \( f \) by
\[
f(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q} \\
-1 & \text{if } x \not\in \mathbb{Q}
\end{cases}.
\]

Then \( f \) is not integrable by the same argument used in Example 7.1.8, but \( |f|(x) = 1 \), and constant functions are certainly integrable. □

**Exercise 7.3.4:** Let \( f \) be continuous on \([a, b]\). For each \( x \in [a, b] \) let \( F(x) = \int_{x}^{b} f \). Show that \( F \) is differentiable and that \( F'(x) = -f(x) \).

*Proof:* Let \( \epsilon > 0 \). Since \( f \) is continuous, there exists \( \delta > 0 \) such that \( |f(t) - f(c)| < \epsilon \) when \( t \in [a, b] \) and \( |t - c| < \delta \). Then for all \( x \in [a, b] \) with \( 0 < |x - c| < \delta \) we have the following:
\[
\left| \frac{F(x) - F(c)}{x - c} + f(c) \right| = \left| \frac{1}{x - c} \left[ \int_{x}^{b} f(t) \, dt - \int_{c}^{b} f(t) \, dt \right] + f(c) \right|
\]

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\[
\left| \frac{1}{x-c} \left( - \int_{c}^{x} f(t) \, dt \right) + \frac{1}{x-c} \int_{c}^{x} f(c) \, dt \right| \\
\leq \left| \frac{1}{x-c} \right| \left| \int_{c}^{x} f(c) \, dt - \int_{c}^{x} f(t) \, dt \right| \\
\leq \left| \frac{1}{x-c} \right| \int_{c}^{x} \left| f(c) - f(t) \right| \, dt \\
< \left| \frac{1}{x-c} \right| \epsilon |x-c| = \epsilon.
\]

So then

\[
\lim_{x \to c} \left| \frac{F(x) - F(c)}{x - c} \right|
\]
does exist, and equals \(-f(c)\). Hence \(F(x)\) is differentiable, and \(F'(x) = -f(x)\). \(\square\)

**Exercise 7.3.10:** Use Theorem 7.3.1 to evaluate \(\lim_{x \to 0} (1/x) \int_{0}^{x} \sqrt{9 + t^2} \, dt\).

**Proof:** Recognize that this limit is the difference quotient representing the derivative of the function \(F(x) = \int_{0}^{x} \sqrt{9 + t^2} \, dt\) at \(x = 0\). Then using this definition of \(F(x)\) we know that

Theorem 7.3.1 tells us that

\[
\lim_{x \to 0} \frac{1}{x} \int_{0}^{x} \sqrt{9 + t^2} \, dt = F'(0) = \sqrt{9 + x^2} \bigg|_{x=0} = \sqrt{9} = 3.
\]

So the limit of interest evaluates to 3. \(\square\)