Geometry v. Gerrymandering: Potential Mathematical Solutions to a Political Problem

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November 7, 2017
Introduction

Gerrymandering
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Introduction
Background
Quantitative Tests
Compactness
Random maps
Efficiency gap
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*Gerrymandering* is the intentional manipulation of election district boundaries to gain political advantage.
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Overview

- Introduction
- Background
  - Examples
  - History
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- Quantitative tests
  - District compactness
  - Random map generation
  - Efficiency gap
Examples

Say we have 15 voters:
Examples

Say we have 15 voters:

- District 1
- District 2
- District 3

Conclusion: Boundaries can profoundly impact outcomes.
Examples

Say we have 15 voters:

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District 2

District 3

Green wins 2-1.
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Orange wins 2-1.
Examples

Say we have 15 voters:

\[\text{District 1} \quad \text{District 2} \quad \text{District 3}\]

Orange wins 2-1.

**Conclusion:** Boundaries can profoundly impact outcomes.
Examples

Republican Congressional Map Used in 2012 and 2014

Hypothetical Democratic Congressional Map

Hypothetical Nonpartisan Congressional Map

Swing Districts

Democratic Districts

Republican Districts

3 10

9 4 0

5 5 3

11 10 12
8 6 9
4 13 2
7 1 3

11 10 12
8 6 9
4 13 2
7 1 3
History

- Governor E. Gerry, 1812
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“Courts ought not to enter this political thicket.”
- Justice F. Frankfurter
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- Davis v. Bandemer, 1986
- Vieth v. Jubelirer, 2004
- LULAC v. Perry, 2006
- Gill v. Whitford, 2017

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Gill v. Whitford:
Current Relevance

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2018: SCOTUS Decision expected in June

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- Plaintiff W. Whitford
A Mathematical Approach?

“If workable standards do emerge to measure these burdens, however, courts should be prepared to order relief.”

- Justice A. Kennedy
Compactness

North Carolina, District 12
Compactness

Florida, District 3
Compactness

Idea:
Measure shape "compactness" by comparing perimeter to the perimeter of a circle with the same area.

\[
\text{circle} \quad A = 1 \quad P \approx 3.145
\]

\[
\text{square} \quad A = 1 \quad P = 4
\]
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\[
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& : P \approx 3.545 \\
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\]
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\[
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\[
\text{rectangle} \quad A = 1 \\
\quad P = 5
\]
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\[ A = 1 \]
\[ P \approx 3.545 \]
\[ A = 1 \]
\[ P = 5 \]
\[ A = 1 \]
\[ P = 8.5 \]
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\[
\begin{align*}
&\text{circle} \\
&A = 1 \\
&P \approx 3.545
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\]

\[
\begin{align*}
&\text{equilateral triangle} \\
&A = 1 \\
&P \approx 4.559
\end{align*}
\]
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**Idea:**
Measure shape “compactness” by comparing perimeter to the perimeter of a circle with the same area.

- **Circle**
  - Area: $A = 1$
  - Perimeter: $P \approx 3.545$

- **Five-pointed star**
  - Area: $A = 1$
  - Perimeter: $P \approx 6.857$
Compactness

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<table>
<thead>
<tr>
<th>Shape</th>
<th>Area</th>
<th>Perimeter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle</td>
<td>$A = 1$</td>
<td>$P \approx 3.545$</td>
</tr>
<tr>
<td>“Snowflake”</td>
<td>$A = 1$</td>
<td>$P \approx 11.475$</td>
</tr>
</tbody>
</table>
Compactness

A troublesome example...
Compactness

A troublesome example...

Louisiana, District 1
Compactness

Strengths:
Compactness

Strengths:
- Intuitively seen
Compactness

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- Straightforward calculation
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- Coincides with notion of gerrymandering
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Weaknesses:
- Problematic near water
- Reasonable compactness standards unclear
- Fails to prevent all gerrymandering

Conclusion:
Simple compactness restrictions merely limit the range of tactics available for drawing unfair maps.
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**Parallel Evolutionary Algorithm for Redistricting**
Random map generation

Useful comparisons can include the following:
Random map generation

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- Compactness
- Competitiveness
- Responsiveness
- Biasedness
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Useful comparisons can include the following:

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- Efficiency gap
Random map generation

2001 plan

2011 plan
Random map generation

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Histograms showing competitiveness for 2001 and 2011 plans.
Random map generation

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2001 plan

2011 plan
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Biasedness

2001 plan

2011 plan

Quantitative Tests

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Random maps

Efficiency gap

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2001 plan

2011 plan

Efficiency Gap

2001 plan

2011 plan
Random map generation

Simulated North Carolinian congressional districts:

- Districts favor Democrats: Democrats win 10-3
- Districts favor Republicans: Republicans win 11-2
- Districts are competitive: Republicans win 7-6
- Districts are compact: Republicans win 9-4
Human geography has a significant effect.
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▶ Data-driven results
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- Leaves "standard" question unanswered
- Very expensive computationally
- "Black box" process

Conclusion:

Random map generation is an outstanding tool for evaluating a given map by comparison.
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“[Partisan symmetry is] widely accepted by scholars as providing a measure of partisan fairness in electoral systems.”
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“Interest in exploring this notion is evident.”
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“[I do not] discount [symmetry’s] utility in redistricting, planning, and litigation.”
- Justice A. Kennedy
The efficiency gap

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In particular, it accounts for “wasted” votes:
- Extra votes for a winning candidate
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For two parties, A and B, the efficiency gap is given as

\[
\text{efficiency gap} = \frac{\text{wasted A votes} - \text{wasted B votes}}{\text{total votes cast}}
\]
The efficiency gap

Say we have 15 voters:

![Image showing orange and green dots to illustrate the efficiency gap]

\[
\text{wasted orange votes} = 4 \\
\text{wasted green votes} = 2 \\
\text{efficiency gap} = \frac{4 - 2}{15} \approx 13.3\%.
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Say we have 15 voters:

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Say we have 15 voters:

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= 1
= 6
\text{efficiency gap} = 1 - 6 = 15 \approx -0.33\%.
The efficiency gap

Say we have 15 voters:

\[
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\text{wasted orange votes} &= 1 \\
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efficiency gap = \frac{1 - 6}{15} \approx -33.3\%
Human geography has a very significant effect on the efficiency gap.

States with an efficiency gap of at least 7%.
The efficiency gap

Strengths:
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Weaknesses:
- Statistically "noisy"
- Misled by poor voter turnout
- Fails to account for human geography

Conclusion:
The efficiency gap is a useful tool, but could be improved with a better understanding of geography.
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“It reminds me a little bit of my steak rub. And so what’s this court supposed to do? A pinch of this, a pinch of that?”
- Justice N. Gorsuch
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The SCOTUS response to this approach has been mixed:

“I think the hard issue in this case is are there standards manageable by a court, not by some group of you know, computer experts?” - Justice S. Breyer
Conclusion

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The SCOTUS response to this approach has been mixed:

“It may be simply my educational background, but I can only describe it as sociological gobbledygook.”

- Chief Justice J. Roberts
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**Conclusion:** We must continue to work towards better tools.
Conclusion

Additional work on the subject:

- Tufts University workshop(s)
- Duke University project
- Application of Markov chains
- Other ideas?
References


